SYDNEY TECHNICAL HIGH SCHOOL

(Est 1911)

MATHEMATICS EXTENSION II

HSC ASSESSMENT TASK 1

MARCH 2003

Time allowed : 70 minutes

Instructions :

- Show all necessary working in every question.
- Start each question on a new page.
- Attempt all questions.
- All questions are of equal value.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- This test forms part of your HSC assessment.
- These questions are to be handed in with your answers.

Name :__________________

Question 1	Question 2	Question 3	Total

MARCH 2003 HSC #1 EXT2.

Question 1 **Marks**

a) Write $\dfrac{3+2i}{1+i}$ in the form $x + iy$ **2**

b) $(3+2i)(d+i)$ is real. Find the value of d **2**

c) In answering questions about the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\ (a>b)$ Ting gave the following answers. **3**

i) eccentricity $0 \le e \le 1$

ii) foci $(0, \pm ae)$

Are Ting's answers correct. Give explanations for your choices.

d) Find $|x+iy+2|$ **2**

e) If z is the complex number $x + iy$ simplify $(z-\bar{z})^2$ **3**

f) i) Sketch the locus of z defined by arg $(z+1)=\dfrac{\pi}{6}$ **2**

ii) Find z in the modulus argument form such that $|z|$ is a minimum **3**

Question 2 **Marks**

a) For the ellipse $4x^2 + 9y^2 = 36$ find

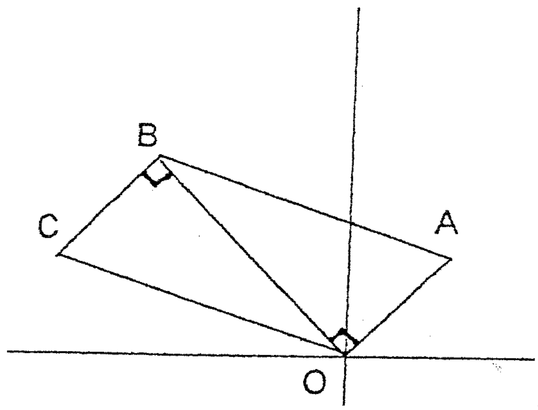
i) eccentricity **2**

ii) co-ordinates of foci **1**

iii) equation of directrices **1**

iv) sketch the ellipse marking all the necessary information **1**

b)



In the diagram $\angle CBO = \angle BOA = 90°$ and $OB = 2OA$. If A is the complex number z

i) Explain why B is the complex number 2iz **2**

ii) If OABC is a parallelogram find the complex number for C **2**

c) Find the equation and sketch the locus of z if $\text{Im}(z^2) = 2$ **2**

d) Let A and B be the complex numbers z_1 and z_2 satisfying $|z_1| = |z_2|$

i) Draw an Argand diagram showing the complex numbers z_1, z_2 and $z_1 + z_2$ (label C), z_1 and z_2 in the first quadrant. **2**

ii) What type of figure is OACB **1**

iii) Mark on your diagram the complex number $z_2 - z_1$ (label D) **1**

iv) Use your diagram or otherwise to show that $\dfrac{z_1 + z_2}{z_2 - z_1}$ is imaginary **2**

Question 3 | **Marks**

a) The complex number z is given by $z = -\sqrt{3} + i$. Find

i) arg z **1**

ii) $|z|$ **1**

iii) z^7 in modulus argument form **2**

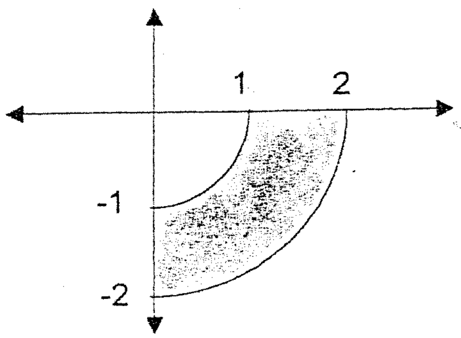
b) The roots of $z^6 - 1 = 0$ are $1, w, w^2, w^3, w^4, w^5$

i) Find w (first complex root) in modulus argument form **1**

ii) Plot all the roots on an Argand diagram **2**

iii) By factorising or otherwise write down the equation whose roots are w, w^3 and w^5 **1**

c) Give the inequalities which describe this region in the complex number plane. **3**



d) i) Prove that the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **3**

at the point $P(x_1\ y_1)$ *is* $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

ii) This ellipse meets the y axis at B *and* B^1. The tangents at B and B^1 to the ellipse meet the tangent at P at Q and Q^1 respectively. **3**

α) Draw a neat sketch labelling each of the points and showing the tangent.

β) Prove $BQ \times B^1Q^1 = a^2$

Question 1

a) $\frac{3+2i}{1+i} \times \frac{1-i}{1-i} = \frac{3-3i+2i+2}{2}$

$= \frac{5}{2} - \frac{i}{2}$

b) $(3+2i)(d+i) = 3d+3i+2di-2$

If real $3+2d = 0$

$d = -3/2$

c).

i) No $0<e<1$

$e=1$ is a parabola

ii) No $(0 \pm ae)$ are the co-ordinates of the focus of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

where $a<b$

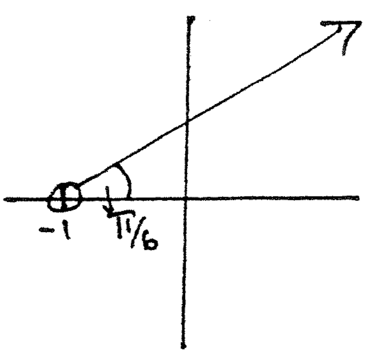
d) $|(x+2)+iy| = \sqrt{(x+2)^2+y^2}$

$(z-\bar{z})^2 = [x+iy-(x-iy)]^2$

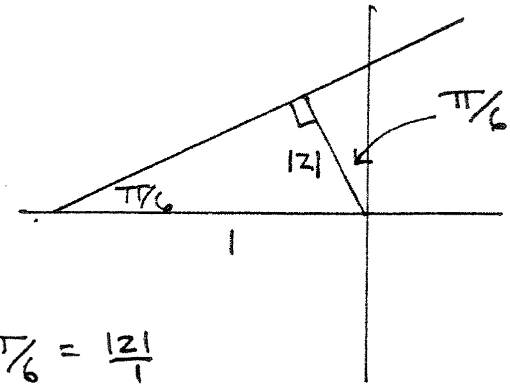
$= [2iy]^2$

$= -4y^2$

) i)



ii)



$\sin\pi/6 = \frac{|z|}{1}$

$\therefore |z| = 1/2$

$z = 1/2 \text{ cis}(\pi/2 + \pi/6)$

$= 1/2 \text{ cis } 2\pi/3$

Question 2

a) $\frac{x^2}{9}+\frac{y^2}{4} = 1$

$ae = \sqrt{9-4}$

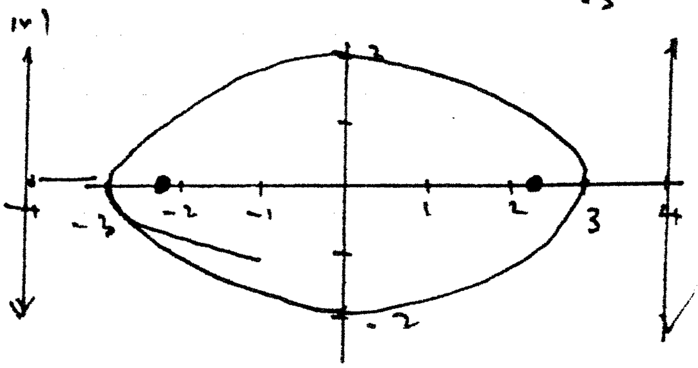
$= \sqrt{5}$

i) $e = \frac{\sqrt{5}}{3}$

ii) focii $= (\pm\sqrt{5}, 0)$

iii) directricies $x = \pm \frac{9}{\sqrt{5}}$

iv)



b) i) $OB = 2OA$ and is rotated through $90°$ in an anticlockwi direction.

ii) $z + \overrightarrow{AB} = 2iz$

$\overrightarrow{AB} = 2iz - z$

$OC \parallel AB$ and $OC = AB$

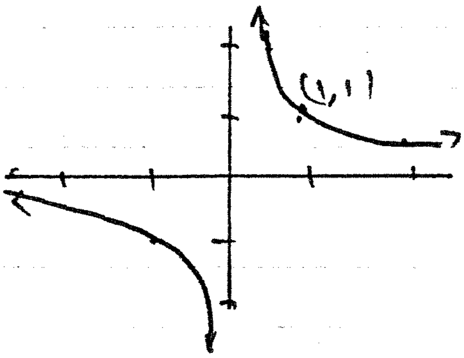
$\therefore \overrightarrow{OC} = 2iz - z$

$\therefore C$ is the complex number $2iz - z$

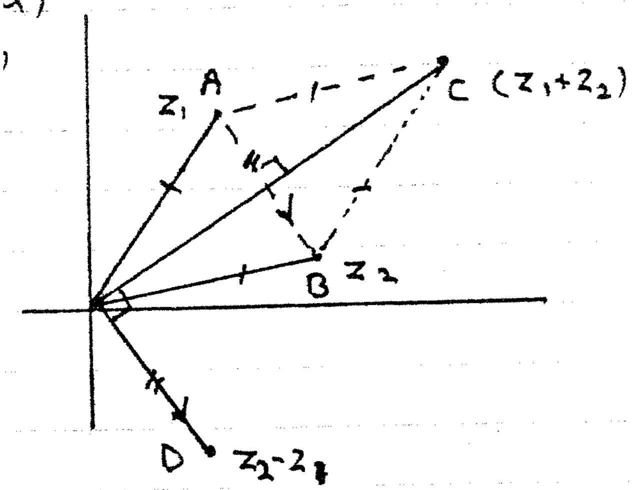
c)

$$z^2 = (x+iy)^2$$
$$= x^2 + 2ixy - y^2$$
$$\therefore \text{Im}(z^2) = 2xy$$

$$\therefore 2xy = 2$$
$$xy = 1$$



d)



i) rhombus

iii)

Now $\arg(z_1+z_2) - \arg(z_2-z_1) = \pm \pi/2$

$\therefore \dfrac{z_1+z_2}{z_2-z_1}$ is imaginary

Question 3

a) i) $\arg z = 5\pi/6$

ii) $|z| = 2$

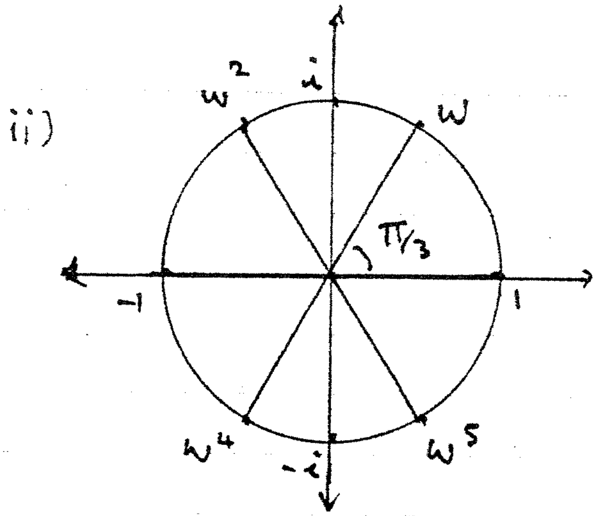
iii) $z^7 = (2 \text{ cis } 5\pi/6)^7$

$$= 128 \text{ cis } \frac{35\pi}{6}$$

$$= 128 \text{ cis } (-\pi/6)$$

b) i) $z = \text{cis } \pi/3$

ii)



iii) $z^6 - 1 = (z^3-1)(z^3+1)$

w, w^3, w^5 are the roots of $z^3+1=0$

c) $1 \le |z| \le 2$, $-\pi/2 \le \arg z \le 0$

d) i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2y\frac{dy}{dx}}{b^2} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{a^2} \times \frac{b^2}{2y}$$

$$= -\frac{b^2x}{a^2y}$$

$$\text{at}(x_1,y_1) = -\frac{b^2x_1}{a^2y_1}$$

Equat of tangent

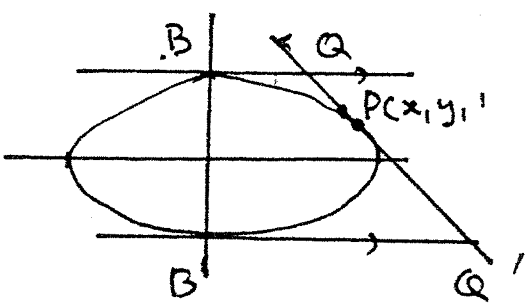
$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$$

$$\frac{yy_1 - y_1^2}{b^2} = \frac{-xx_1 + x_1^2}{a^2}$$

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

but (x_1,y_1) lies on ellipse

$$\therefore \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$



Equation of tangent at B

$y = b$

at Q $\quad \dfrac{xx_1}{a^2} + \dfrac{by_1}{b^2} = 1$

$$x = \left(1 - \frac{by_1}{b^2}\right)\frac{a^2}{x_1}$$

$$= \frac{a^2b^2 - a^2by_1}{b^2x_1}$$

Likewise at Q'

$$x = \frac{a^2b^2 + a^2by_1}{b^2x_1}$$

Now $BQ \times B'Q'$

$$= \left(\frac{a^2b^2 - a^2by_1}{b^2x_1}\right)\left(\frac{a^2b^2 + a^2by_1}{b^2x_1}\right)$$

$$= \frac{a^4b^4 - a^4b^2y_1^2}{b^4x_1^2}$$

$$= \frac{a^2(a^2b^2 - a^2y_1^2)}{b^2x_1^2}$$

Now since (x_1,y_1) lies on the curve

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

i.e. $b^2x_1^2 + a^2y_1^2 = a^2b^2$

$$\therefore \frac{a^2(a^2b^2 - a^2y_1^2)}{a^2b^2 - a^2y_1^2}$$

$$= a^2$$

$\therefore$ Proven